$\hat{\pi}_k$: Optimum value of π_k

$$\hat{\pi}_k = \arg\max_{\pi_k} \Big[\underbrace{\ln p(b|s^2) + \eta \sum_k \pi_k - \eta}_{\ell(\pi)} \Big]$$

constraint $\sum_k \pi_k = 1$
use Lagrange multiplier.

$$\ln p(b|s^2) = \ln \prod_{j=1}^{P} \int p(b_j | w_j, s^2)\, p(w_j)\, dw_j$$

$$= \sum_{j=1}^{P} \ln \left(\sum_{k=1}^{K} \pi_k \phi_{jk} \right)$$

$$\frac{\partial}{\partial \pi_k} \ell(\pi) = \frac{\partial}{\partial \pi_k} \left[\sum_{j=1}^{P} \ln \left(\sum_{k=1}^{K} \pi_k \phi_{jk} \right) + \eta \sum_k \pi_k - \eta \right]$$

$$= \sum_{j=1}^{P} \frac{\phi_{jk}}{\sum_k \pi_k \phi_{jk}} + \eta = \frac{1}{\pi_k} \sum_{j=1}^{P} \frac{\pi_k \phi_{jk}}{\sum_k \pi_k \phi_{jk}} + \eta = 0 \quad \text{(at optimum)} \qquad (1)$$

There is no closed form solution, but we can cheat a little and assume $\sum_{j=1}^{P} \frac{\pi_k \phi_{jk}}{\sum_k \pi_k \phi_{jk}}$ is constant, which yields

$$\hat{\pi}_k = -\frac{1}{\eta} \sum_{j=1}^{P} \frac{\pi_k \phi_{jk}}{\sum_k \pi_k \phi_{jk}}$$

Summing over k in Eq. (1) gives $\eta = -P$

$$\Rightarrow \hat{\pi}_k = +\frac{1}{P} \sum_{j=1}^{P} \frac{\pi_k \phi_{jk}}{\sum_k \pi_k \phi_{jk}}.$$

▨ Point mass at λ_k. ~~Prior~~

Prior. $p_k(w) = \prod_{j=1}^{P} p_k(w_j) = \prod_{j=1}^{P} \delta(w_j - \lambda_k) = \delta(w - \mathbb{I}\lambda_k)$

Posterior. $p_k(w_j | b_j, s^2) = \dfrac{p(b_j | w_j, s^2)}{p(b_j | \lambda_k, s^2)} \delta(w_j - \lambda_k) = \dfrac{p(b_j | w_j, s^2)\, p_k(w_j)}{\phi_{jk}}$

$\phi_{jk} := p(b_j | \lambda_k, s^2) = \mathcal{N}(b_j | 0, s^2 \lambda_k)$

▨ Mixture of point mass.

Prior. $p(w_j) = \sum_{k=1}^{K} \pi_k\, p_k(w_j) = \sum_{k=1}^{K} \pi_k\, \delta(w_j - \lambda_k)$

Posterior. $p(w_j | b_j, s^2) = \dfrac{p(b_j | w_j, s^2)\, p(w_j)}{\int p(b_j | w_j, s^2)\, p(w_j)\, dw_j} = \dfrac{p(b_j | w_j, s^2)\, p(w_j)}{\phi_j}$

$$= \frac{\sum_{k=1}^{K} \pi_k\, p_k(w_j)\, p(b_j | w_j, s^2)\, \phi_{jk} / \phi_{jk}}{\phi_j}$$

$\phi_j := \int p(b_j | s^2)$

$$= \frac{\sum_{k=1}^{K} \pi_k\, \phi_{jk}\, p_k(w_j | b_j, s^2)}{\phi_j} = \sum_{k=1}^{K} \pi_k'\, p_k(w_j | b_j, s^2),$$

where $\pi_k' = \dfrac{\pi_k\, \phi_{jk}}{\phi_j}$

Now, $\phi_j = \int p(b_j | w_j, s^2) \sum_k \pi_k\, p_k(w_j)\, dw_j$

$$= \sum_k \pi_k\, p(b_j | \lambda_k, s^2) = \sum_k \pi_k\, \phi_{jk} \Rightarrow \boxed{\pi_k' = \frac{\pi_k\, \phi_{jk}}{\sum_k \pi_k\, \phi_{jk}}}$$

We can now compute the expectation of any function of w_j:

$$\mathbb{E}[f(w_j)] = \int f(w_j)\, p(w_j | b_j, s^2)\, dw_j = \sum_{k=1}^{K} \pi_k' \int f(w_j)\, p_k(w_j | b_j, s^2)\, dw_j$$

Interestingly this is true for any mixture prior

Here we used the definition of $p_k(w_j | b_j, s^2)$ from point-mass prior.

$$= \sum_{k=1}^{K} \pi_k'\, f(\lambda_k)$$